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On Transitive Actions of compact Lie groups on Stein space

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ABSTRACT

Our main aim is to extend results of P. Heinzener on complexification to Stein space. We derive some theorems and corollary regarding actions of compact Lie group on Stein manifold. If an equivariant complexification exists in the sense of Levy, then it may be applicable on any other kind of complexification. A real analytic G-principal bundle is a real analytic manifold X with a proper G-action.

INTRODUCTION

Let *K* be a compact Lie group and *X* a Stein K-space. Let us denote by X//K quotient of X with respect to the algebra $O(X)^K$ of K-invariant holomorphic functions. The quotient map is denoted by $\pi_x : X \to X // K$. The quotient X//K is a Stein space whose structure sheaf is given by the presheaf $U \to O\left(\pi_x^{-1}(U)\right)^K$. Let B(x) denote the smallest analytic K-subset of X which contains a given point $x \in X$ and *E(x)* the intersection over all K-invariant analytic K-subsets of $\pi_x^{-1}(\pi_x(x))$. The analytic *K-set E(x)* depends only on the point *p* = $\pi_x(x) \in X // K$ and is non-empty. If $\varphi : X \to [0, \infty)$ is a K-invariant differentiable strictly-plurisubharmonic exhaustion function on *X*, then we set:

$$M_\phi = \{x \in X;\ \phi|B(x) \text{ has a minimal value in } x\} \qquad \ldots(1)$$

The K-subset M_ϕ *is* closed in *X*. To see this, note that K^C acts on *X* in the infinitesimal sense. For an element v of the Lie algebra $k^C = k \otimes C$ of K^C, let us denote by $\tilde{v}$ the induced vector field on X.

$$M_\phi = \{x \in X;\ \tilde{v}\ (\phi)\ (x) = 0 \text{ for all } v \in K^C\} \qquad \ldots(2)$$

It shows that M_ϕ *is* closed.

Lemma:

The natural map $M_\phi \to X // K$ is proper and the induced map $M_\phi / K \to X // K$ is an isomorphism of topological spaces.

Proof:

For $r \in$ R, let $D_\phi(r)$ let us denote the open K-subset of $x \in X$ such that $\varphi(x) < r$. P. Heinzener proved the following statements

(i) $\pi_x\left(D_\phi(r)\right) = \pi_x\left(D_\phi(r) \cap M_\phi\right)$ is open in $X // K$

(ii) for every $x \in X$ one has $E(x) \cap M_\phi = K$ x_o for some $x_o \in M_\phi$,

(iii) for every $x \in M_\phi$ one has $E(x) = B(x)$.

It is known that X is an orbit convex subset of its complexification P. Heinzener derived some results on complexifiction. We extend it to Stein manifold.

Theorem:

Let K be a compact Lie group and X a Stein-K manifold of finite K-orbit type. Let ω : X $\rightarrow$ X be a K-equivariant anti holomorphic involution on X with a non-empty set X^ω of ω fixed points. Then the natural map $X^\omega \rightarrow X // K$ is proper and induces a closed topological embedding $X^\omega / K \rightarrow X // K$.

Proof:

Let us assume X^ω as a closed subset of some M_ϕ, then we can derive its complexification. Let X^C be the complexification of the K-space X. Since $X^C = K^C$. X, the holomorphic Stein K^C -space X^C is of finite K-orbit type. Hence, there exists a holomorphic linearly equivariant embedding $f : X^C \rightarrow C^n$.. The map g : X $\rightarrow$ C^{2n}, defined by $g(x) = (f(x), \overline{f(\omega(x))})$ is a linearly equivariant holomorphic immersion and $g(X^\omega)$ is contained in the totally real K-subspace V = { (z, w) $\in C_n \times C_n$. $w = z$ } of C^{2n}. Let <- , - > be a positive definite Hermitian K-invariant product on C^{2n} which is an extension of a scalar product on V. Then $V \subset M_\delta$ for the function δ : $C^{2n} \rightarrow$ R, δ(z) = <z, z>, we obtain $X^\omega \subset M_\phi$ with the function $\varphi = \delta \circ g$.

It has been assumed here that a holomorphic Stein K^C-manifold X may be embedded linearly equivariant into some C^N if and only if the K-orbit type of X is finite. Which is established by the formulation of the statement in terms of K^C -orbit type. Which yield only a sufficient condition.

Corollary:

If K is a compact subgroup of a Lie group G, then the image H of K^C in G^C is closed and H is the complexification of the image $\overline{K}$ of K in G^C. The holomorphic G^C-space G^C/H is a holomorphic G^C-complexification of G/K and a holomorphic G^C-extension of i(G)/$\overline{K}$. Moreover, i(G)/$\overline{K}$ is a closed totally real submanifold of the Stein manifold G^C/H. In particular, if G is a holomorphically extendable Lie group, then G^C/K^C is a holomorphic G^C -extension of G/K.

Proof:

If ϕ is a non-negative differentiable strictly plurisubharmonic K- invariant function on a complex K-space X, then the set of zeros of φ has a basis of open Stein K-neighborhoods in X. The trivial K-action and smooth X is given by Harvev and Well's with only minor changes. In particular, under the assumptions of the theorem (1.3), we obtain the following.

Theorem :

Identifying X^ω (resp. X^ω/K) with its image in X (resp. X//K), it

follows that:

(i) X^{ω} has a basis of open Stein K-neighborhoods in X, and

(ii) X^{ω} */K* has a basis of open Stein neighborhoods in *X//K.*

Proof:

The quotient X^{ω} */K* is a closed subset of *X//K* and *X//K* can be identified in a natural way with $X^C//K^C$. It shows that X^{ω} /K is a closed subset of *V/K* which is a closed subset of $V^C//K^C$, $V^C = C^{2n}$. Hence we identify $V^C//K^C$ with a closed subset of some C^q such that *V/K* becomes a closed subset of R^q. If $y_1, \ldots, y_q$ denote the imaginary parts of the coordinates $z_1, \ldots, z_q$ of C_q, then φ $(z_1, \ldots, z_q) = y_1^2 + \ldots\ldots + y_q^2$ defines a strictly plurisubharmonic function on every open neighborhood of X^{ω} */K* in *X//K* which vanishes on X^{ω} */K*. This proves part (ii). Hence, the theorem is proved.

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